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On various solutions of the coupled KP equation

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Abstract

The coupled KP (cKP) equation possesses N-soliton solutions with many more degrees of freedom than the solitons for the usual KP equation have. In comparison, cKP solutions can therefore be expected to model far more complex interactions than their KP counterparts. In this paper we present some typical solutions for the cKP system (and for some of its reductions: a complex coupled KdV equation and a coupled Boussinesq equation) and we analyse their interaction and asymptotic properties.

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1. Introduction

The coupled Kadomtsev–Petviashvili (cKP) equation was first proposed as the soliton system which arises from coupling the Kadomtsev–Petviashvili (KP) to the Davey–Stewartson equation [1]:

$$\left(D_x^4 - 4D_xD_t + 3D_y^2\right)\tau\cdot\tau = 24\hat{\sigma}\sigma \tag{1}$$

$$\left(D_x^3 + 2D_t - 3D_xD_y\right)\hat{\sigma}\cdot\tau = 0 \tag{2}$$

$$\left(D_x^3 + 2D_t + 3D_xD_y\right)\sigma\cdot\tau = 0. \tag{3}$$

The above equations are written in terms of the (Hirota) bilinear operators D_x, D_y and D_t, which can be defined through

$$D_t^kD_y^mD_x^n f\cdot g = \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^k\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^m\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^n f(x,y,t)g(x',y',t')\Big|_{\substack{x'=x\\ y'=y\\ t'=t}}. \tag{4}$$

Equations (1)–(3) can be transformed into the following coupled nonlinear partial differential equations:

$$(4u_t - 6uu_x - u_{xxx})_x - 3u_{yy} + 24(v\hat{v})_{xx} = 0 \tag{5}$$

$$2v_t + 3uv_x + v_{xxx} + 3\left(v_{xy} + v\int^x u_y\,\mathrm{d}x\right) = 0 \tag{6}$$

$$2\hat{v}_t + 3u\hat{v}_x + \hat{v}_{xxx} - 3\left(\hat{v}_{xy} + \hat{v}\int^x u_y\,\mathrm{d}x\right) = 0 \tag{7}$$

by means of the dependent variable transformations:

$$u = 2(\log\tau)_{xx} \tag{8}$$

$$v = \sigma/\tau \qquad \hat{v} = \hat{\sigma}/\tau. \tag{9}$$

The cKP equation has *N*-soliton solutions which can be expressed in terms of Pfaffians [2]. Furthermore, it is known that the cKP equation is actually part of a hierarchy of integrable equations—the bilinear identity of which was first derived in [3]—and that the functions τ, σ and $\hat{\sigma}$ appearing in the system (1)–(3) can therefore be regarded as genuine tau functions in the sense of Sato theory [4, 5]. Although after these initial discoveries this integrable system lay dormant for nearly a decade, recently there has been a renewed interest not only in the cKP equation itself but also in its related hierarchy and possible discretizations thereof [6–8]. Special interest was found in the algebraic background of this hierarchy and in its possible applications to the theory of random matrices [9, 10].

Considering that in this respect the understanding of the cKP equation and its properties has considerably widened, it is striking that no discussion of its solutions (or of their characteristics) seems to exist in the literature. We would like to argue, however, that such a discussion is more than worthwhile (besides being long overdue) as it allows one to uncover a host of peculiar solitonic interactions. For it is immediately clear that as a consequence of their Pfaffian origin, the *N*-soliton solutions of the cKP equation are characterized by many more parameters than would be the case for comparable solutions of the (usual) KP equation. Consequently, the number of degrees of freedom which can be used to generate various types of interactions between such solutions is also much larger. Hence the hope to find solutions which will, qualitatively, behave very differently in comparison to those of the KP equation.

In this paper we discuss various examples of solutions of the cKP equation—amongst which are resonant solitons—and we compare their interaction properties with similar solutions for the KP equation. We shall also discuss solutions of two equations which are obtained from the cKP equation by simple reductions: one is the so-called complex coupled Korteweg–deVries (ccKdV) equation [11] and the other can best be described as a coupled Boussinesq (cB) equation. In section 2, the functional forms of the one- and two-soliton solutions of the cKP equation are listed and the ccKdV equation and the cB equations are introduced. In section 3, we discuss the solutions of the ccKdV equation and we show that it possesses rather peculiar periodic solutions. The solutions of the cB equation are discussed in section 4, where it will be shown that the cB equation possesses resonant soliton solutions which exhibit more freedom than those of the good-Boussinesq equation [12]. Finally, in section 5 we discuss the soliton solutions of the cKP equation itself. We show that the cKP equation has resonant solitons which—again due to an increased number of degrees of freedom—allow for more complex interaction than would be possible in the case of the KP equation [13].

2. The cKP equation and its reductions

2.1. The soliton solutions of the cKP equation

In the following, p_j and q_j $(j = 1, 2, \ldots, 2N)$ will represent arbitrary parameters which characterize the behaviour of the solitons; α_j, β_j $(j = 1, 3, \ldots, 2N-1)$ are (equally arbitrary) parameters which will play the role of so-called phase constants for the individual solitons (i.e. fixing the relative positions of the solitons). We define the phase functions ξ_p and the so-called interaction factors $\mathrm{A}(p_i, p_j; q_k, q_l)$ by

$$\xi_p := px + p^2 y + p^3 t \tag{10}$$

$$\theta(p_j, p_k) := \xi_{p_j} + \xi_{p_k} \tag{11}$$

$$\mathrm{A}(p_i, p_j; q_k, q_l) := \frac{(p_i - p_j)(q_k - q_l)}{(p_i - q_k)(p_i - q_l)(p_j - q_k)(p_j - q_l)}. \tag{12}$$

For example, the one-soliton solution to the cKP system (5)–(7) can be represented in terms of the following tau functions:

$$\tau = 1 + \alpha_1\beta_1\mathrm{A}(p_1, p_2; q_1, q_2)\,\mathrm{e}^{\theta(p_1,p_2)-\theta(q_1,q_2)} \tag{13}$$

$$\hat{\sigma} = \alpha_1(p_1 - p_2)\,\mathrm{e}^{\theta(p_1,p_2)} \tag{14}$$

$$\sigma = \beta_1(q_1 - q_2)\,\mathrm{e}^{-\theta(q_1,q_2)} \tag{15}$$

using the dependent variable transformations (8) and (9). However, in the following we would like to refer to these tau functions themselves as 'one-soliton' solutions (and to subsequent expressions for such tau functions as '*N*-solitons'). We do so, of course, only for the sake of brevity: what is really meant by this slight abuse of language is that through the dependent variable transformation (8), the tau function τ gives rise to a one-soliton (or, in general an *N*-soliton) solution in the field u (and similarly that the tau functions σ and $\hat{\sigma}$ do so for the fields v and $\hat{v}$).

Thus we say that the two-soliton solution for the cKP system is given by

$$\begin{aligned}\tau = 1 &+ \alpha_1\beta_1\mathrm{A}(p_1, p_2; q_1, q_2)\,\mathrm{e}^{\theta(p_1,p_2)-\theta(q_1,q_2)} + \alpha_3\beta_3\mathrm{A}(p_3, p_4; q_3, q_4)\,\mathrm{e}^{\theta(p_3,p_4)-\theta(q_3,q_4)}\\ &+ \alpha_1\beta_3\mathrm{A}(p_1, p_2; q_3, q_4)\mathrm{e}^{\theta(p_1,p_2)-\theta(q_3,q_4)} + \alpha_3\beta_1\mathrm{A}(p_3, p_4; q_1, q_2)\,\mathrm{e}^{\theta(p_3,p_4)-\theta(q_1,q_2)}\\ &+ \alpha_1\beta_1\alpha_3\beta_3\frac{\prod_{1\leqslant i<j\leqslant 4}\{(p_i - p_j)(q_i - q_j)\}}{\prod_{i,j=1}^{4}(p_i - q_j)}\mathrm{e}^{\theta(p_1,p_2)+\theta(p_3,p_4)-\theta(q_1,q_2)-\theta(q_3,q_4)}\end{aligned} \tag{16}$$

$$\begin{aligned}\hat{\sigma} = \alpha_1(p_1 - p_2)\,\mathrm{e}^{\theta(p_1,p_2)} &+ \alpha_2(p_3 - p_4)\,\mathrm{e}^{\theta(p_3,p_4)}\\ &+ \alpha_1\alpha_3\beta_1\frac{(q_1 - q_2)\prod_{1\leqslant i<j\leqslant 4}(p_i - p_j)}{\prod_{1\leqslant i\leqslant 4,\ j=1,2}(p_i - q_j)}\,\mathrm{e}^{\theta(p_1,p_2)+\theta(p_3,p_4)-\theta(q_1,q_2)}\\ &+ \alpha_1\alpha_3\beta_3\frac{(q_3 - q_4)\prod_{1\leqslant i<j\leqslant 4}(p_i - p_j)}{\prod_{1\leqslant i\leqslant 4,\ j=3,4}(p_i - q_j)}\,\mathrm{e}^{\theta(p_1,p_2)+\theta(p_3,p_4)-\theta(q_3,q_4)}\end{aligned} \tag{17}$$

$$\begin{aligned}\sigma = \beta_1(q_1 - q_2)\,\mathrm{e}^{-\theta(q_1,q_2)} &+ \beta_2(q_3 - q_4)\,\mathrm{e}^{-\theta(q_3,q_4)}\\ &+ \alpha_1\beta_1\beta_3\frac{(p_1 - p_2)\prod_{1\leqslant i<j\leqslant 4}(q_i - q_j)}{\prod_{i=1,2,\ 1\leqslant j\leqslant 4}(p_i - q_j)}\,\mathrm{e}^{\theta(p_1,p_2)-\theta(q_1,q_2)-\theta(q_3,q_4)}\\ &+ \alpha_3\beta_1\beta_3\frac{(p_3 - p_4)\prod_{1\leqslant i<j\leqslant 4}(q_i - q_j)}{\prod_{i=3,4,\ 1\leqslant j\leqslant 4}(p_i - q_j)}\,\mathrm{e}^{\theta(p_3,p_4)-\theta(q_1,q_2)-\theta(q_3,q_4)}.\end{aligned} \tag{18}$$

Although we shall not give its explicit form here, we would like to remark that in general the N-soliton solution of the cKP equation will depend on $4N$ parameters and $2N$ phase constants, whereas in the case of the KP equation the N-soliton solution only depends on $2N$ parameters and N phase constants.

2.2. The complex coupled KdV equation

The complex coupled KdV (ccKdV) equation is obtained from the cKP equation by selecting y-independent tau functions τ, σ and $\hat{\sigma}$ and in addition to this, by requiring σ and $\hat{\sigma}$ to be complex conjugate, i.e. $\bar{\sigma} = \hat{\sigma}$. In fact, under this reduction τ itself becomes a real function. The bilinear form of the ccKdV equation takes the form

$$\left(D_x^4 - 4D_x D_t\right) \tau \cdot \tau = 24|\sigma|^2 \tag{19}$$

$$\left(D_x^3 + 2D_t\right) \sigma \cdot \tau = 0. \tag{20}$$

By setting $u = 2(\log \tau)_{xx}$ and $v = \sigma/\tau$, equations (19) and (20) are turned into the following coupled partial differential equations:

$$(4u_t - 6uu_x - u_{xxx})_x + 24(|v|^2)_{xx} = 0 \tag{21}$$

$$2v_t + 3uv_x + v_{xxx} = 0. \tag{22}$$

Note that the well-known coupled KdV equation [14] is obtained from the cKP equation by a similar restriction to y-independent tau functions but for a different constraint: $\sigma = \hat{\sigma}$.

2.3. The coupled Boussinesq system

We obtain a new coupled system from the cKP equation by restricting the tau functions to t-independent ones and by (again) requiring complex conjugacy for the auxiliary tau functions σ and $\hat{\sigma}$ $(\bar{\sigma} = \hat{\sigma})$. As in the previous case, τ will become a real function under this reduction and we call the resulting system the *coupled Boussinesq equation* (cB). Note that in order to realize the constraint $\bar{\sigma} = \hat{\sigma}$ one needs to transform the independent variable y as $y \to \mathrm{i}y$. The bilinear form of the cB equation then reduces to

$$\left(D_x^4 - 3D_y^2\right) \tau \cdot \tau = 24|\sigma|^2 \tag{23}$$

$$\left(D_x^3 - 3\,\mathrm{i}D_x D_y\right) \sigma \cdot \tau = 0. \tag{24}$$

Equations (23) and (24) can be transformed into the following coupled partial differential equations (by setting $u = 2(\log \tau)_{xx}$, $v = \sigma/\tau$):

$$(-6uu_x - u_{xxx})_x + 3u_{yy} + 24(|v|^2)_{xx} = 0 \tag{25}$$

$$3uv_x + v_{xxx} - 3\mathrm{i}\left(v_{xy} + v\int^x u_y \,\mathrm{d}x\right) = 0. \tag{26}$$

3. Solutions of the ccKdV equation

The N-soliton solution for the ccKdV equation is obtained by imposing the constraints

$$p_{j+1} = \mathrm{i}p_j \qquad q_{j+1} = \mathrm{i}q_j \tag{27}$$

on the parameters in the N-soliton solution of the cKP equation, followed by a series of specializations of the remaining parameters:

$$1. \begin{cases} p_j \to p_j(1-\mathrm{i})/2 \\ q_j \to q_j(1-\mathrm{i})/2 \end{cases} \tag{28}$$

$$2. \begin{cases} p_j = a_j + \mathrm{i}b_j \\ q_j = -a_j + \mathrm{i}b_j = -\bar{p}_j \\ \beta_j = \bar{\alpha}_j \end{cases} \tag{29}$$

where $j = 1, 3, \ldots, 2N-1$ and a_j, b_j are real constants. Then, without loss of generality, we can further simplify the expressions for the solutions by transforming the phase constants α_j in the following way:

$$-\mathrm{i}\alpha_j p_j \to \alpha_j. \tag{30}$$

Here we give the explicit form of the resulting one- and two-soliton solutions, expressed in terms of the phase functions and interaction factors:

$$\eta_j = 2a_j\left(x - \left({a_j}^2 - 3{b_j}^2\right)t/2\right) \tag{31}$$

$$A_j = \frac{1}{2{a_j}^2\left({a_j}^2 - {b_j}^2\right)}. \tag{32}$$

The one-soliton solution takes the form

$$\tau = 1 + |\alpha_1|^2 A_1\,\mathrm{e}^{\eta_1} \tag{33}$$

$$\hat{\sigma} = \alpha_1\,\mathrm{e}^{(a_1+\mathrm{i}b_1)x-(a_1+\mathrm{i}b_1)^3t/2} \tag{34}$$

and the τ-part of the two-soliton solution is

$$\begin{aligned}\tau = 1 + |\alpha_1|^2 A_1\,\mathrm{e}^{\eta_1} + |\alpha_3|^2 A_3\,\mathrm{e}^{\eta_3} + 2\mathrm{Re}\{\alpha_1\bar{\alpha}_3(\mu+\mathrm{i}\nu)\,\mathrm{e}^{\mathrm{i}\varphi}\}\,\mathrm{e}^{\frac{1}{2}(\eta_1+\eta_3)} + |\alpha_1|^2|\alpha_3|^2 A_1 A_3 \\ \times\frac{\{(a_1-a_3)^2+(b_1-b_3)^2\}^2}{\{(a_1+a_3)^2+(b_1-b_3)^2\}^2} \\ \times\frac{\left\{\left({a_1}^2-{b_1}^2+{a_3}^2-{b_3}^2\right)^2+(2a_1b_1+2a_3b_3)^2\right\}}{\left\{\left({a_1}^2-{b_1}^2+{a_3}^2-{b_3}^2\right)^2+(2a_1b_1-2a_3b_3)^2\right\}}\times\mathrm{e}^{\eta_1+\eta_3}\end{aligned} \tag{35}$$

$$\varphi := (b_1-b_3)x + \left(-3{a_1}^2b_1 + {b_1}^3 + 3{a_3}^2b_3 - {b_3}^3\right)\frac{t}{2} \tag{36}$$

where μ and ν are rational expressions of a_j and b_j $(j = 1, 3)$ defined (respectively) as the real and imaginary parts of $\mathrm{A}(p_1, p_2; q_3, q_4)$ under condition (27) and transformations (28) and (29). For reasons of brevity we do not present the explicit form of the auxiliary tau function σ.

In general, the N-soliton solution of the ccKdV equation will depend on $2N$ real parameters a_j, b_j and N phase constants α_j for $j = 1, 3, \ldots, 2N-1$. At this point it is important to note that if one or more of the interaction factors A_j should happen to be negative, the field u will become divergent. However, such divergences can easily be avoided by requiring that the parameters satisfy the inequality

$${a_j}^2 - {b_j}^2 > 0. \tag{37}$$

For example, choosing the parameters as

$$b_j = \pm \frac{a_j}{\sqrt{3}} \qquad (j = 1, 3, \ldots, 2N-1) \tag{38}$$

we obtain a rather interesting solution for the ccKdV equation. In this case the phase functions η_j appearing in the exponential terms in the *N*-soliton solution no longer depend on the variable *t*, but the solution still has an overall *t*-dependence in the form of trigonometric functions multiplying some of the exponential terms (for example, in the case of (35) such a *t*-dependence enters through the function φ). Hence, one may conclude that these solitons will be essentially stationary, were it not for possible interactions between them. In particular, by controlling the phase constants, the solitons can be prepared such that they exhibit periodic collisions (see figure 1 for such a two-soliton interaction; please note that this figure—as well as all subsequent plots of actual solutions—depicts the 'main' solitonic field *u*).

We would like to point out that due to the particular structure of the soliton interaction factors (32), resonant solutions (describing fission or fusion processes of solitons) cannot exist in the present case. However, we will show that such solutions do exist for the coupled Boussinesq equation.

4. Solutions of the cB equation

As after the reduction to the cB equation only the independent variables *x* and *y* remain, we choose *y* as the 'time' variable according to which the various solutions evolve (as opposed to the 'natural' variable *t* in the ccKdV case). The *N*-soliton solution for the cB equation is obtained by transforming this time variable *y* and by specializing the parameters in the solutions in the following way:

$$y \to \mathrm{i}y \tag{39}$$

$$p_j = a_j + \mathrm{i}b_j \tag{40}$$

$$p_{j+1} = \frac{a_j + \sqrt{3}b_j}{2} + \mathrm{i}\frac{-\sqrt{3}a_j + b_j}{2} \tag{41}$$

$$q_j = -a_j + \mathrm{i}b_j = -\bar{p}_j \tag{42}$$

$$q_{j+1} = -\frac{a_j + \sqrt{3}b_j}{2} + \mathrm{i}\frac{-\sqrt{3}a_j + b_j}{2} = -\bar{p}_{j+1} \tag{43}$$

$$\beta_j = -\bar{\alpha}_j \tag{44}$$

where $j = 1, 3, \ldots, 2N-1$ and a_j, b_j are real constants. Without loss of generality we can simplify the explicit forms of the solutions by transforming the phase constants α_j as

$$\alpha_j(p_j - p_{j+1}) \longrightarrow \alpha_j. \tag{45}$$

In the following we shall (again) only give the explicit forms of the one- and two-soliton solutions. First we introduce

$$\begin{aligned}\hat{\eta}_j &= (3a_j + \sqrt{3}b_j)x + \left(\sqrt{3}{a_j}^2 - 2a_jb_j - \sqrt{3}{b_j}^2\right)y \\ &= \sqrt{3}(\sqrt{3}a_j + b_j)\left\{x + \frac{1}{\sqrt{3}}(a_j - \sqrt{3}b_j)y\right\}\end{aligned} \tag{46}$$

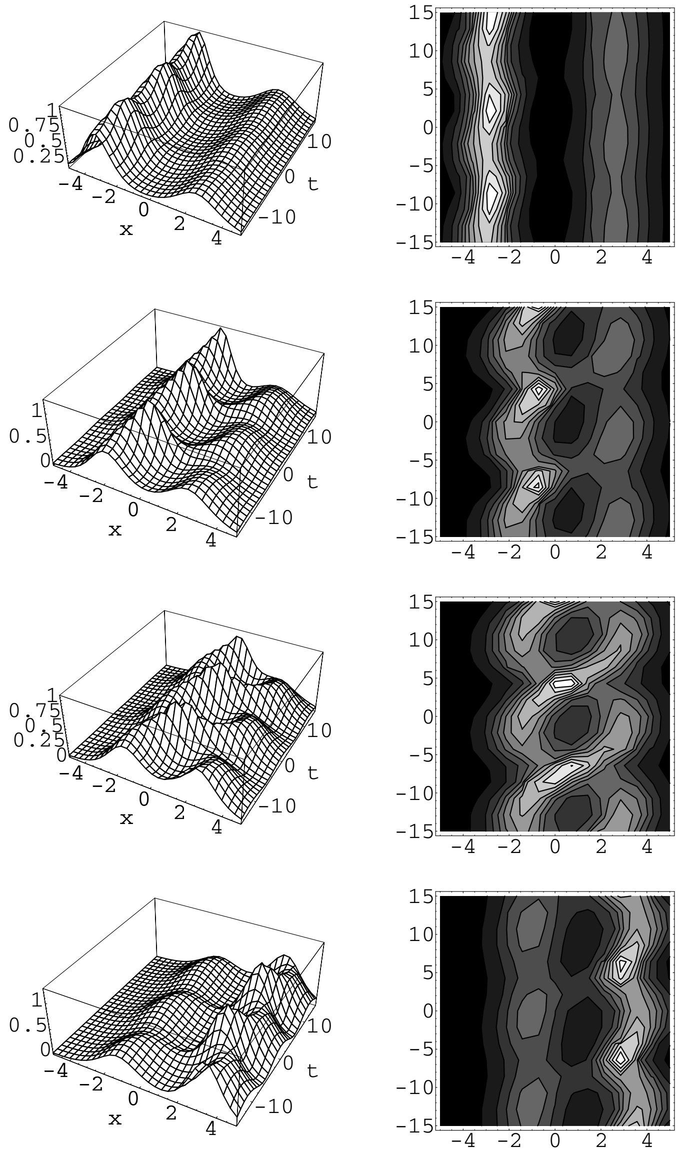

Figure 1. An example of a periodic soliton solution for the ccKdV equation, $(a_1, b_1, a_3, b_3) = \left(1, \frac{1}{\sqrt{3}}, \frac{2}{3}, \frac{2}{3\sqrt{3}}\right)$. From the top, $(\alpha_1, \alpha_3) = (20, 1)$, $(4, 1)$, $(2, 1)$ and $(1/2, 1)$.

$$\hat{A}_j = \frac{1}{2a_j(a_j + \sqrt{3}b_j)(\sqrt{3}a_j + b_j)^2}. \tag{47}$$

The one-soliton solution can then be expressed as

$$\tau = 1 + |\alpha_1|^2 \hat{A}_1 \, \mathrm{e}^{\hat{\eta}_1} \tag{48}$$

$$\sigma = \bar{\alpha}_1 \, \mathrm{e}^{\frac{1}{2}\eta_1} \, \mathrm{e}^{\mathrm{i}\{\frac{\sqrt{3}}{2}(a_1 - \sqrt{3}b_1)x - \frac{1}{2}({a_1}^2 + 2\sqrt{3}a_1 b_1 - {b_1}^2)y\}} \tag{49}$$

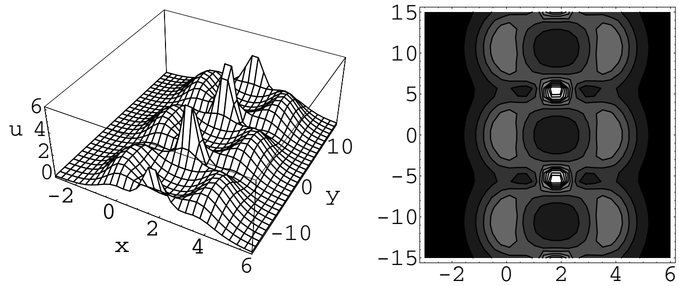

Figure 2. An example of a periodic soliton solution for the cB equation, $(a_1, b_1, a_3, b_3) = (\frac{5}{6}, \frac{5}{6\sqrt{3}}, \frac{1}{2}, \frac{1}{2\sqrt{3}})$, $(\alpha_1, \alpha_3) = (1, 1.2)$.

and the τ-part for the two-soliton solution takes the form

$$\begin{aligned}\tau = 1 &+ |\alpha_1|^2 \hat{A}_1\, \mathrm{e}^{\hat{\eta}_1} + |\alpha_3|^2 \hat{A}_3\, \mathrm{e}^{\hat{\eta}_3} + \mathrm{Re}\{\alpha_1\bar{\alpha}_3(\hat{\mu} + \mathrm{i}\hat{\nu})\, \mathrm{e}^{\mathrm{i}\phi}\}\, \mathrm{e}^{\frac{1}{2}(\hat{\eta}_1+\hat{\eta}_3)} \\ &+\alpha_1|^2|\alpha_3|^2 \hat{A}_1\hat{A}_3 \times \frac{(a_1/2 + \sqrt{3}b_1/2 - a_3)^2 + (-\sqrt{3}a_1/2 + b_1/2 - b_3)^2}{(a_1/2 + \sqrt{3}b_1/2 + a_3)^2 + (-\sqrt{3}a_1/2 + b_1/2 - b_3)^2} \\ &\times \frac{(a_1 - a_3)^2 + (b_1 - b_3)^2}{(a_1 + a_3)^2 + (b_1 - b_3)^2} \times \frac{(a_1 - a_3/2 - \sqrt{3}b_3/2)^2 + (b_1 + \sqrt{3}a_3/2 - b_3/2)^2}{(b_1 + \sqrt{3}a_3/2 - b_3/2)^2 + (a_1 + a_3/2 + \sqrt{3}b_3/2)^2} \\ &\times \frac{(a_1/2 + \sqrt{3}b_1/2 - a_3/2 - \sqrt{3}b_3/2)^2 + (-\sqrt{3}a_1/2 + b_1/2 + \sqrt{3}a_3/2 - b_3/2)^2}{(-\sqrt{3}a_1/2 + b_1/2 + \sqrt{3}a_3/2 - b_3/2)^2 + (a_1/2 + \sqrt{3}b_1/2 + a_3/2 + \sqrt{3}b_3/2)^2} \\ &\times \mathrm{e}^{\hat{\eta}_1+\hat{\eta}_3} \end{aligned} \tag{50}$$

$$\begin{aligned}\phi := \left(-\frac{\sqrt{3}}{2}a_1 + \frac{3}{2}b_1 + \frac{\sqrt{3}}{2}a_3 - \frac{3}{2}b_3\right)x + \bigg(&\frac{1}{2}{a_1}^2 + \sqrt{3}a_1b_1 - \frac{1}{2}{b_1}^2 \\ &- \frac{1}{2}{a_3}^2 - \sqrt{3}a_3b_3 + \frac{1}{2}{b_3}^2\bigg)y\end{aligned} \tag{51}$$

where $\hat{\mu}$, $\hat{\nu}$ are rational expressions of the parameters a_j and b_j $(j = 1, 3)$, respectively, defined as the real part and the imaginary part of $A(p_1, p_2; q_3, q_4)$ after the reduction (40)–(43). Again, we choose not to present the explicit forms of the auxiliary tau function σ.

The cB *N*-soliton solution will, in general, depend on $2N$ real parameters a_j, b_j and N phase constants α_j $(j = 1, 3, \dots, 2N - 1)$. As was previously remarked for the ccKdV equation, note that if one or more of the coefficients $\hat{A}_j$ are negative the corresponding field u will become divergent. In order to avoid such divergences, we should choose the parameters such that

$$a_j(a_j + \sqrt{3}b_j) > 0. \tag{52}$$

Moreover, if we fix the parameters b_j by

$$b_j = \frac{a_j}{\sqrt{3}} \qquad (j = 1, 3, \dots, 2N - 1) \tag{53}$$

the phases of the corresponding *N*-soliton solutions will no longer depend on the y variable. The resulting solutions are very similar to those discussed in the case of the ccKdV equation (see figure 2).

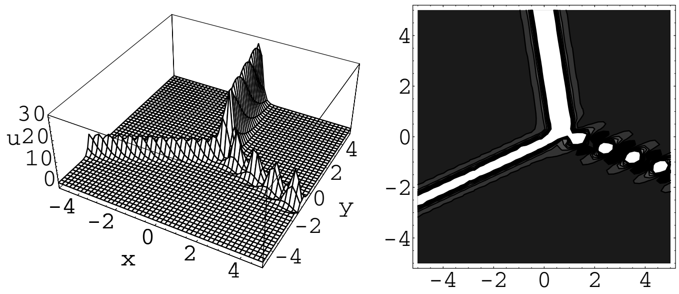

Figure 3. An example of a resonant solution of the cB equation, under condition (54), $(a_1, b_1; \alpha_1, \alpha_3) = (2, 1; 1, 1)$.

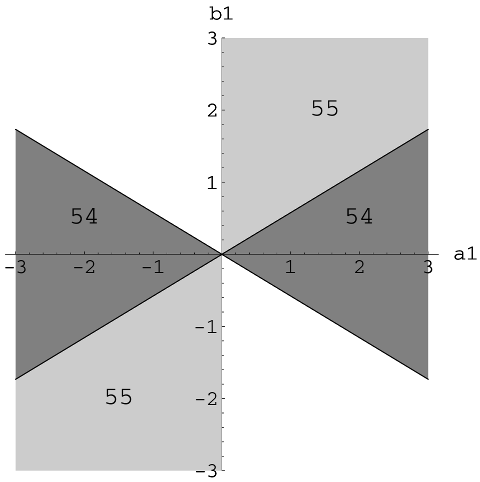

Figure 4. The regions in which the interaction factors are positive, depending on the specific condition ((54) or (55)) the parameters satisfy. The two diagonal lines are $b_1 = \pm\frac{1}{\sqrt{3}}a_1$.

The cB equation has also resonant solutions. First, let us consider resonances which are obtained from two-soliton interactions (see figure 3). In order to achieve a resonance, we have to choose one set of parameters a_j and b_j such that they satisfy one of the following conditions, here expressed on the set of parameters (a_3, b_3):

$$\begin{cases} a_3 = (a_1 - \sqrt{3}b_1)/2 \\ b_3 = (\sqrt{3}a_1 + b_1)/2 \end{cases} \tag{54}$$

or

$$\begin{cases} a_3 = (a_1 + \sqrt{3}b_1)/2 \\ b_3 = (-\sqrt{3}a_1 + b_1)/2. \end{cases} \tag{55}$$

In this case—because of these specific relations—the coefficient of the last term in expression (50) will vanish and a resonance occurs. Next, in order to avoid possible divergences in the field u, we need to impose certain restrictions on the parameter space (a_1, b_1) (see figure 4). Asymptotic analysis then shows that the solutions obtained in this way describe the interaction (fusion or fission) of two resonant solitons.

Here we only present the asymptotic behaviour in the case of (54). If we define ϵ to be the sign of $K := \frac{a_1}{b_1}\left(3{a_1}^2 - 5{b_1}^2\right)$, the phase functions η_1', η_2', η_4' as

$$\eta_1' := x + \frac{1}{\sqrt{3}}(a_1 - \sqrt{3}b_1)y \tag{56}$$

$$\eta_2' := x - \frac{1}{\sqrt{3}}(a_1 + \sqrt{3}b_1)y \tag{57}$$

$$\eta_4' := x + \frac{1}{b_1}\left({a_1}^2 - {b_1}^2\right)y \tag{58}$$

and the function $B := 2\,\mathrm{Re}\{\alpha_1\bar{\alpha}_3(\hat{\mu} + \mathrm{i}\hat{\nu})\,\mathrm{e}^{\mathrm{i}\phi}\}$ (this function does not affect the asymptotics) we find the following asymptotic behaviour for τ:

$$\begin{aligned}
&y \to +\infty : \tau \sim 1 + \hat{A}_1\,\mathrm{e}^{(3a_1+\sqrt{3}b_1)\eta_1'} \\
&y \to -\infty : \tau \sim 1 + \hat{A}_3\,\mathrm{e}^{(3a_1-\sqrt{3}b_1)\eta_2'} \\
&y \to -\epsilon\infty : \tau \sim 1 + \frac{B}{\hat{A}_3}\,\mathrm{e}^{\sqrt{3}b_1\eta_4'} + \frac{\hat{A}_1}{\hat{A}_3}\,\mathrm{e}^{2\sqrt{3}b_1\eta_4'}.
\end{aligned}$$

Note that this resonant solution depends on two parameters a_1, b_1 which—as mentioned above—can be chosen freely, within certain bounds in order to avoid divergences in the field u. This freedom could for example be used to control the amplitude as well as the velocity of one of the solitons. This is in stark contrast with the case of the good Boussinesq equation for which there is only one such free parameter [12].

We can also obtain resonant solutions from a three-soliton solution. Obviously, by imposing conditions (54) or (55) on a three-soliton solution for the cB equation we can construct a solution where two out of three solitons exhibit mutual resonances, leaving one 'free' soliton. It is therefore natural to ask whether or not, by making some left-over terms vanish in the tau function τ, it would be possible to obtain a 'multi-resonance', i.e. a resonant solution which has three independent phases in, for example, the $y \to -\infty$ direction, but only one phase in the opposite direction $y \to +\infty$.

As far as such possible extra 'vanishing' terms are concerned, there are essentially three nontrivial choices (i.e. choices for which the number of solitons is not reduced, as would, for example, be the case if $a_1 = a_3 = a_5$, $b_1 = b_3 = b_5$). These are

$$\begin{cases} a_3 = (a_1 - \sqrt{3}b_1)/2 \\ b_3 = (\sqrt{3}a_1 + b_1)/2 \\ a_5 = (a_1 + \sqrt{3}b_1)/2 \\ b_5 = (-\sqrt{3}a_1 + b_1)/2 \end{cases} \tag{59}$$

$$\begin{cases} a_3 = (a_1 - \sqrt{3}b_1)/2 \\ b_3 = (\sqrt{3}a_1 + b_1)/2 \\ a_5 = (-a_1 - \sqrt{3}b_1)/2 \\ b_5 = (\sqrt{3}a_1 - b_1)/2 \end{cases} \tag{60}$$

$$\begin{cases} a_3 = (a_1 + \sqrt{3}b_1)/2 \\ b_3 = (-\sqrt{3}a_1 + b_1)/2 \\ a_5 = (-a_1 + \sqrt{3}b_1)/2 \\ b_5 = (-\sqrt{3}a_1 - b_1)/2. \end{cases} \tag{61}$$

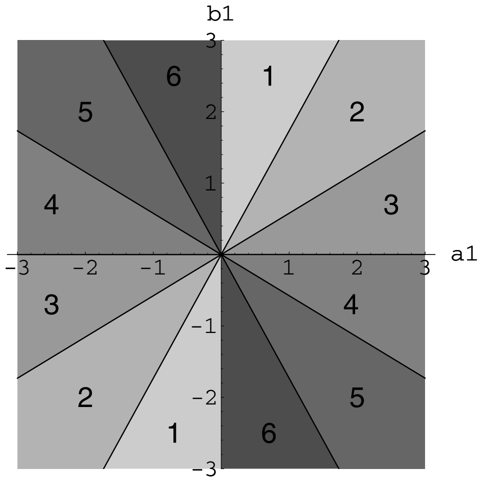

Figure 5. Parameter regions 1 to 6: the diagonal lines depict $b_1 = \pm\sqrt{3}a_1$ and $b_1 = \pm\frac{1}{\sqrt{3}}a_1$.

Under condition (59), for example, the tau function τ becomes

$$\tau = 1 + \tilde{A}_1\,\mathrm{e}^{\zeta_1} + \tilde{A}_2\,\mathrm{e}^{\zeta_2} + \tilde{A}_3\,\mathrm{e}^{\zeta_3} + B_1\,\mathrm{e}^{\frac{1}{2}(\zeta_1+\zeta_2)} + B_2\,\mathrm{e}^{\frac{1}{2}(\zeta_1+\zeta_3)} + B_3\,\mathrm{e}^{\frac{1}{2}(\zeta_2+\zeta_3)} + \tilde{A}_2\tilde{A}_3C\,\mathrm{e}^{\zeta_2+\zeta_3} \tag{62}$$

$$\zeta_1 = (3a_1 + \sqrt{3}b_1)x + \left(\sqrt{3}{a_1}^2 - 2a_1b_1 - \sqrt{3}{b_1}^2\right)y \tag{63}$$

$$\zeta_2 = (3a_1 - \sqrt{3}b_1)x + \left(-\sqrt{3}{a_1}^2 - 2a_1b_1 + \sqrt{3}{b_1}^2\right)y \tag{64}$$

$$\zeta_3 = 2\sqrt{3}b_1x + 4a_1b_1y \tag{65}$$

$$\tilde{A}_1 = \frac{|\alpha_1|^2}{2a_1(a_1 + \sqrt{3}b_1)(\sqrt{3}a_1 + b_1)^2} \tag{66}$$

$$\tilde{A}_2 = \frac{|\alpha_3|^2}{2a_1(-\sqrt{3}a_1 + b_1)^2(a_1 - \sqrt{3}b_1)} \tag{67}$$

$$\tilde{A}_3 = \frac{|\alpha_5|^2}{4{b_1}^2(a_1 + \sqrt{3}b_1)(-a_1 + \sqrt{3}b_1)}. \tag{68}$$

As before, the functions B_j consist of factors which only contain rational expressions of a_1, b_1 and trigonometric functions (and hence they do not influence the overall asymptotic behaviour of the solitons). The constant C is a positive rational expression in a_1 and b_1. First of all, note that one of the three coefficients $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ is always negative (for all values of the parameters (a_1, b_1)) and that, furthermore, it can be shown that in the other two cases as well (conditions (60) or (61)) τ will always depend on similar coefficients $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3$. Hence, a potentially multi-resonant solution u corresponding to such a tau function would always become divergent, whatever the values of the parameters. Secondly and more importantly, however, careful analysis of the asymptotic behaviour of the above tau function shows that it is not really a 'multi-resonant solution' after all. In order to see this and as the asymptotic

behaviour differs for different values of the parameters, we have to distinguish between six different cases or parameter regions (see figure 5). Hereafter, we omit the explicit coefficients of the exponential terms in these solutions and instead concentrate on the specific asymptotics for each of them:
region 1:

$$y \to +\infty : \tau \sim \begin{cases} e^{\zeta_3}(1 + e^{\zeta_2}) \\ e^{\zeta_2}(1 + e^{\zeta_3}) \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} 1 + e^{\zeta_1} \\ 1 + e^{\zeta_2} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_3)} + e^{\zeta_1 - \zeta_3} \end{cases}$$

region 2:

$$y \to +\infty : \tau \sim \begin{cases} e^{\zeta_3}(1 + e^{\zeta_2}) \\ 1 + e^{\zeta_3} \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} 1 + e^{\zeta_2} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_2)} + e^{\zeta_1 - \zeta_2} \end{cases}$$

region 3:

$$y \to +\infty : \tau \sim \begin{cases} 1 + e^{\zeta_3} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_3)} + e^{\zeta_1 - \zeta_3} \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} 1 + e^{\zeta_2} \\ e^{\zeta_2}(1 + e^{\zeta_3}) \end{cases}$$

region 4:

$$y \to +\infty : \tau \sim \begin{cases} 1 + e^{\zeta_1} \\ 1 + e^{\zeta_3} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_2)} + e^{\zeta_1 - \zeta_2} \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} e^{\zeta_3}(1 + e^{\zeta_2}) \\ e^{\zeta_2}(1 + e^{\zeta_3}) \end{cases}$$

region 5:

$$y \to +\infty : \tau \sim \begin{cases} 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_2)} + e^{\zeta_1 - \zeta_2} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_3)} + e^{\zeta_1 - \zeta_3} \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} e^{\zeta_3}(1 + e^{\zeta_2}) \\ e^{\zeta_2}(1 + e^{\zeta_3}) \end{cases}$$

region 6:

$$y \to +\infty : \tau \sim \begin{cases} e^{\zeta_3}(1 + e^{\zeta_2}) \\ e^{\zeta_2}(1 + e^{\zeta_3}) \end{cases}$$

$$y \to -\infty : \tau \sim \begin{cases} 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_2)} + e^{\zeta_1 - \zeta_2} \\ 1 + e^{\frac{1}{2}(\zeta_1 - \zeta_3)} + e^{\zeta_1 - \zeta_3}. \end{cases}$$

From the above analysis it is clear that it is impossible to obtain a truly multi-resonant soliton solution. Instead, however, we see an interesting phenomenon appearing in regions 5 and 6, where completely different pairs of phases (i.e. 'solitons') exist in the $y \to +\infty$ and $y \to -\infty$ directions.

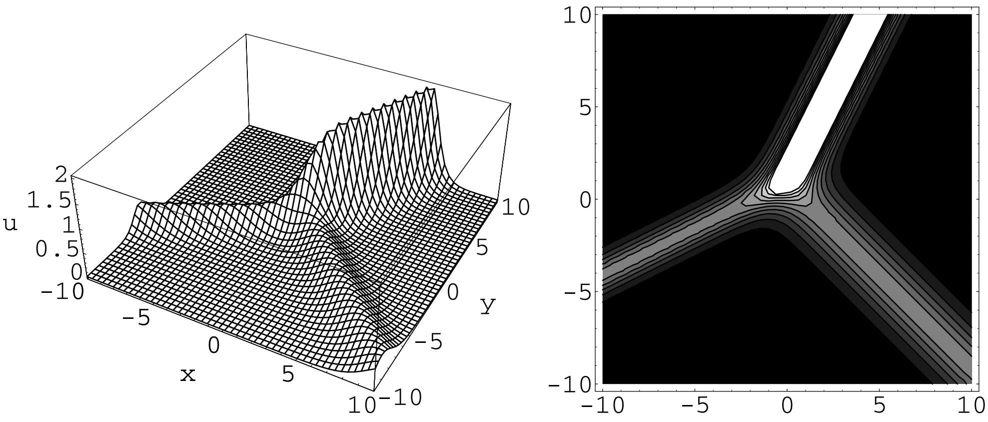

Figure 6. An example of a resonant solution of the cKP equation, depicted at $t = 0$. The two (free) phases were chosen as : $x + y + \frac{1}{2}t$ and $2x - y - 2t$.

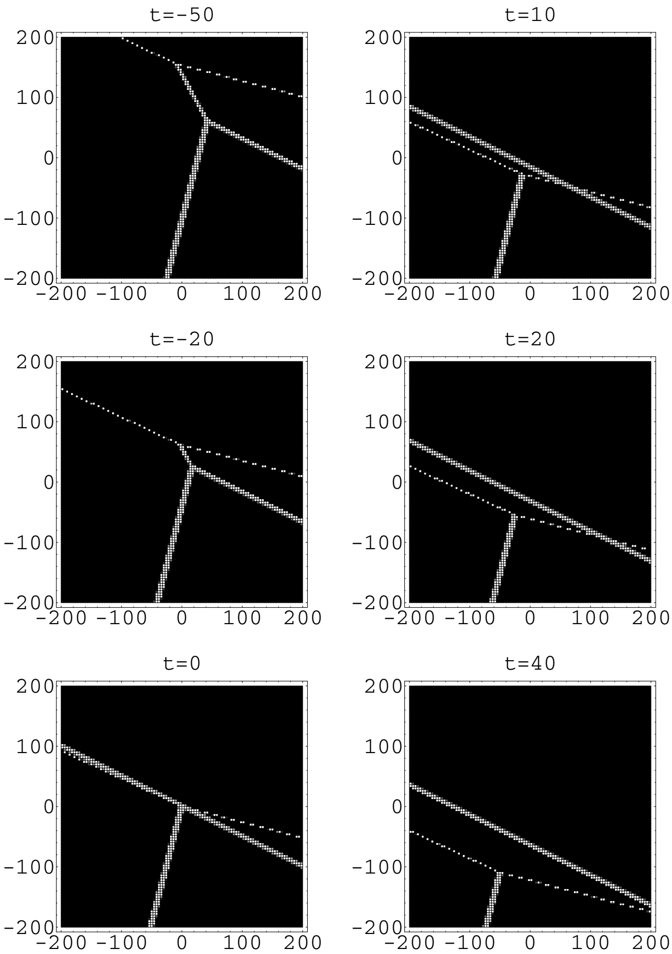

Figure 7. An example of a resonant solution of the cKP equation: contour plot in the x–y plane. $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4; \alpha_1, \alpha_3, \beta_1, \beta_3) = (3, 2, 4/3, 1, 3/2, -1, 1/2, -1; 1, 1, 1, 1)$.

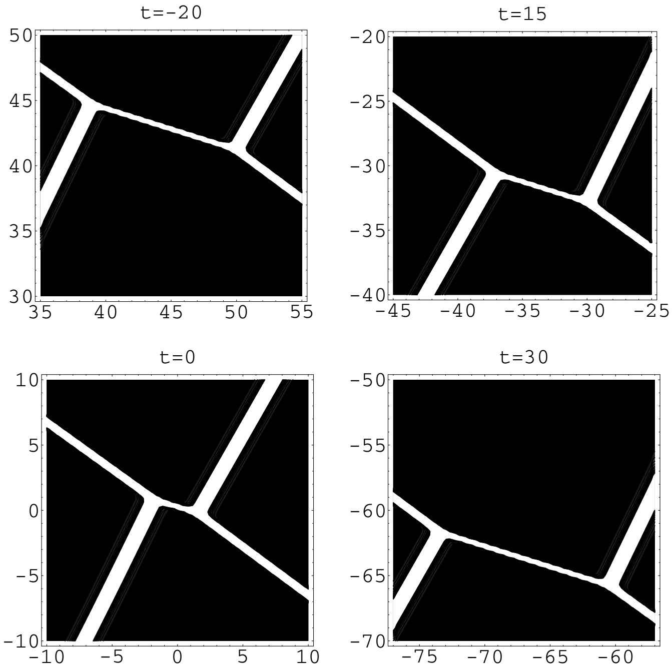

Figure 8. An example of a two-soliton solution of the cKP equation: contour plot in the x–y plane. Note that the ranges in subsequent plots are the same although positions of the windows vary. $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = (3, 3/2, -1/2, -3/2, 2/3, 1/2, -2/3, -4/3)$, $(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1)$.

5. Solutions for the cKP equation

In section 2 we have already presented the explicit forms of the one- and two-soliton solutions of the cKP equation. Here we should first discuss the conditions one has to impose on their parameters in order to avoid divergences in the dependent variable u. As before, such divergences would result from the appearance of at least one negative term in the tau function τ. A necessary and sufficient condition which allows one to avoid all divergences can (in principle) be given but it turns out to be very complex and we shall therefore omit it here. We present a sufficient condition instead: if we choose parameters which satisfy the inequality

$$q_{2N} \leqslant q_{2N-1} \leqslant \cdots \leqslant q_1 < p_{2N} \leqslant p_{2N-1} \leqslant \cdots \leqslant p_1 \tag{69}$$

all coefficients in τ become non-negative and the variable u will always be bounded. Note that this choice makes all factors in the denominators of the respective coefficients (12) positive and all factors in the numerators of these coefficients non-negative.

Next, let us consider resonant solutions obtained from two-soliton solutions of the cKP equation. There exist two different types of such resonant solutions. A first type is obtained by choosing parameters which satisfy one of the following conditions:

$$p_1 = p_2 \tag{70}$$

$$p_3 = p_4 \tag{71}$$

$$q_1 = q_2 \tag{72}$$

$$q_3 = q_4. \tag{73}$$

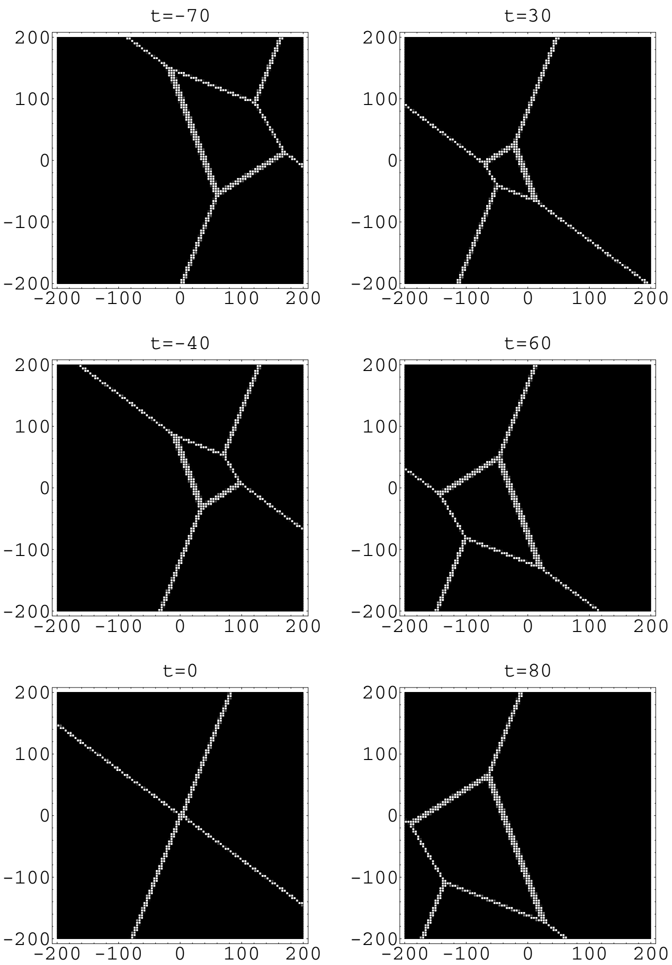

Figure 9. An example of a tetragonal two-soliton solution for the cKP equation: contour plot in the x–y plane. $(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4) = (2, 3/2, 1, 1/2, -1/3, -1/2, -1, -4/3)$, $(\alpha_1, \alpha_3, \beta_1, \beta_3) = (1, 1, 1, 1)$.

As an example we consider the case of (71). If $p_3 = p_4$, τ becomes

$$\tau = 1 + \alpha_1\beta_1 A(p_1, p_2; q_1, q_2)\,\mathrm{e}^{\theta(p_1,p_2)-\theta(q_1,q_2)} + \alpha_1\beta_3 A(p_1, p_2; q_3, q_4)\,\mathrm{e}^{\theta(p_1,p_2)-\theta(q_3,q_4)} \quad (74)$$

and (as expected) it corresponds to a resonant solution which describes the interaction of the two solitons (see figure 6). Note that in this τ six free parameters $(p_1, p_2, q_1, q_2, q_3, q_4)$ remain. This means that one has complete control over two of the phases. In comparison, in the case of the KP equation the resonant solution obtained from a generic two-soliton solution only has three free parameters (see, e.g. [15] for a survey of results concerning this topic).

Another type of resonant solution can be obtained by choosing parameters which satisfy one of the following conditions:

$$p_1 = p_3 \quad (75)$$

$$p_1 = p_4 \quad (76)$$

$$p_2 = p_3 \quad (77)$$

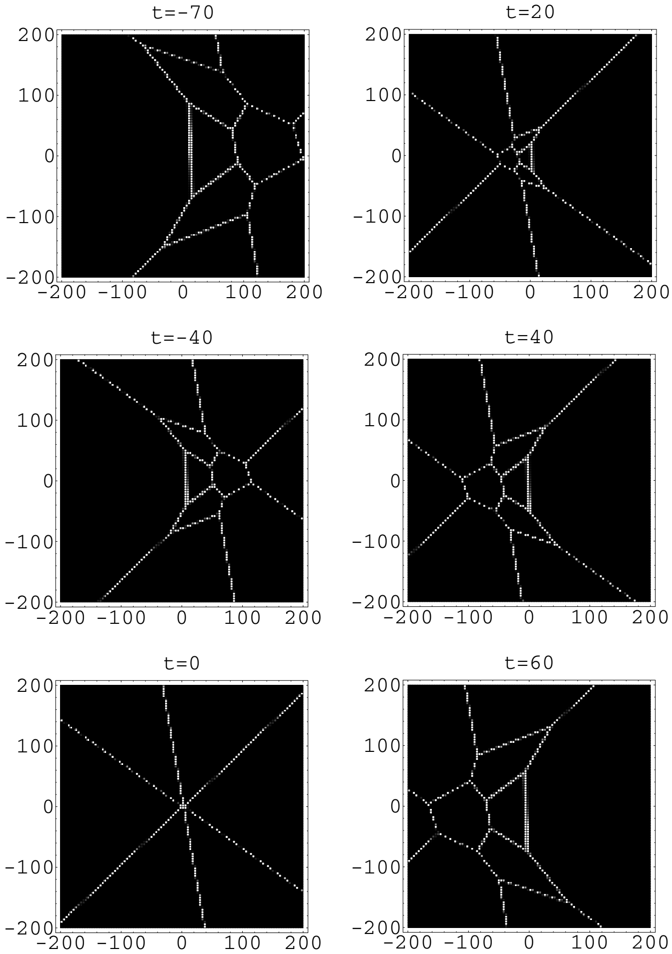

Figure 10. An example of a 'spider web'-like three-soliton solution for the cKP equation: contour plot in the x–y plane. $(p_1, p_2, p_3, p_4, p_5, p_6) = (2, 3/2, 4/3, 1, 1/2, 1/3)$, $(q_1, q_2, q_3, q_4, q_5, q_6) = (-1/4, -1/2, -1, -101/100, -6/5, -5/3)$, $(\alpha_1, \alpha_3, \alpha_5, \beta_1, \beta_3, \beta_5) = (1, 1, 1, 1, 1, 1)$.

$$p_2 = p_4 \tag{78}$$

$$q_1 = q_3 \tag{79}$$

$$q_1 = q_4 \tag{80}$$

$$q_2 = q_3 \tag{81}$$

$$q_2 = q_4. \tag{82}$$

Let us for example consider case (82) for which the function τ becomes

$$\tau = 1 + \alpha_1\beta_1 A(p_1, p_2; q_1, q_2)\, e^{\theta(p_1,p_2)-\theta(q_1,q_2)} + \alpha_3\beta_3 A(p_3, p_4; q_3, q_2)\, e^{\theta(p_3,p_4)-\theta(q_3,q_2)} + \alpha_1\beta_3 A(p_1, p_2; q_3, q_2)\, e^{\theta(p_1,p_2)-\theta(q_3,q_2)} + \alpha_3\beta_1 A(p_3, p_4; q_1, q_2)\, e^{\theta(p_3,p_4)-\theta(q_1,q_2)}. \tag{83}$$

Whereas for the KP equation the time evolution of a resonant solution is nothing but a mere ‘parallel displacement’ of the resonance pattern, solution (83) (see figure 7) exhibits a time evolution which is far more complex.

The same is true for ordinary soliton interactions. Although the standard KP evolution only allows for parallel displacement of the soliton interactions patterns, cKP solitons have considerably more freedom built-in to them. For example in figure 8, the length of the intermediate wave which is created in the interaction can be seen changing over time. Furthermore, for a different choice of parameters the interaction states can be organized so as to form a tetragon (as in the case of figure 9). This specific pattern arises when one chooses the parameters such that some terms become very small while remaining nonzero, i.e. such that it is very close to a resonant state. Figure 10 shows the effect of a similar choice of parameters on a three-soliton solution. We can see hexagons and several more branches appearing, resulting in a ‘spider web’-like pattern. It is expected that these patterns will become more and more intricate as the number of solitons increases.

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